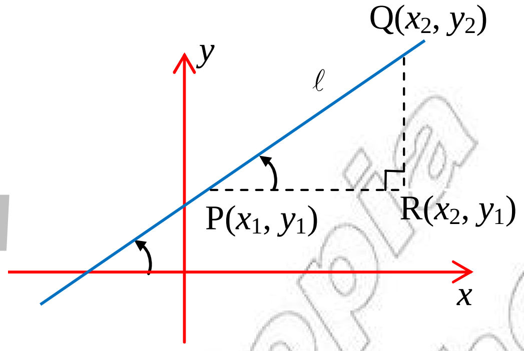
Unit 4



COORDINATE GEOMETRY

Unit Outcomes:

After completing this unit, you should be able to:

- *apply the distance formula to find the distance between any two given points in the coordinate plane.*
- *formulate and apply the section formula to find a point that divides a given line segment in a given ratio.*
- *write different forms of equations of a line and understand related terms.*
- *describe parallel or perpendicular lines in terms of their slopes.*

Main Contents

INTRODUCTION

IN UNIT 3, YOU HAVE SEEN AN IMPORTANT CONNECTION BETWEEN ALGEBRA AND GEOME
THE GREAT DISCOVERIES OF 17TH CENTURY MATHEMATICS WAS THE SUBJECT OF ***analytic geometry***. IT IS OFTEN REFERRED TO AS CARTESIAN GEOMETRY, WHICH IS THE METHOD OF STUDYING GEOMETRY BY MEANS OF A COORDINATE SYSTEM AND ASSOCIATED ALG

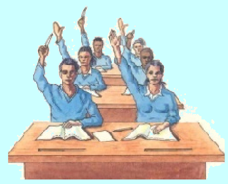
IN ANALYTIC GEOMETRY, WE DESCRIBE PROPERTIES OF GEOMETRIC FIGURES SUCH AS CIRCLES, ETC., IN TERMS OF ORDERED PAIRS AND EQUATIONS.

4.1 DISTANCE BETWEEN TWO POINTS

IN GRADE 9, YOU HAVE DISCUSSED THE NUMBER LINE AND YOU HAVE SEEN THAT THERE TO-ONE CORRESPONDENCE BETWEEN THE SET OF REAL NUMBERS AND THE SET OF P NUMBER LINE. YOU HAVE ALSO SEEN HOW TO LOCATE A POINT IN THE COORDINATE P REMEMBER THE FACT THAT THERE IS A ONE-TO-ONE CORRESPONDENCE BETWEEN THE THE PLANE AND THE SET OF ALL ORDERED PAIRS OF REAL NUMBERS?

THE FOLLOWING ACTIVITY WILL HELP YOU TO REVIEW THE FACTS YOU DISCUSSED IN GRADE 9.

ACTIVITY 4.1



1 CONSIDER THE NUMBER LINE GIVEN IN FIGURE 4.1.

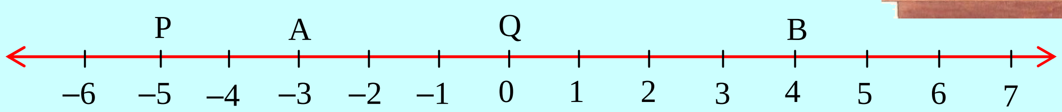


Figure 4.1

A FIND THE COORDINATES OF POINTS P, Q, A AND B.

B FIND THE DISTANCE BETWEEN POINTS

I P AND Q **II** Q AND B **III** P AND B

2 ON A NUMBER LINE, THE TWO POINTS P AND Q HAVE COORDINATES x_1 AND x_2.

A FIND THE DISTANCE BETWEEN P AND Q OR PQ.

B FIND THE DISTANCE BETWEEN Q AND P.

C DISCUSS THE RELATIONSHIP BETWEEN YOUR ANSWERS IN A AND B.

D DISCUSS THE RELATIONSHIP BETWEEN $|x_2 - x_1|$ AND $|x_1 - x_2|$.

3 HOW DO YOU PLOT THE COORDINATES OF POINTS IN THE COORDINATE PLANE?

4 WHAT ARE THE COORDINATES OF THE ORIGIN OF THE xy-PLANE?

5 DRAW A COORDINATE PLANE AND PLOT THE FOLLOWING POINTS.

P (3,–4), Q (–3,–2), R (–2, 0), S (4, 0), T (2, 3), U (–4, 5) AND V (0, 0).

6 THE POSITION OF EACH POINT ON THE COORDINATE PLANE IS DETERMINED BY ITS (PAIR OF NUMBERS.

A WHAT IS THE x-COORDINATE OF A POINT ON THE y-AXIS?

B WHAT IS THE y-COORDINATE OF A POINT ON THE x-AXIS?

7 LET P (2, 3) AND Q (2, 8) BE POINTS ON THE COORDINATE PLANE.

A PLOT THE POINTS P AND Q.

B IS THE LINE THROUGH POINTS P AND Q VERTICAL OR HORIZONTAL?

C WHAT IS THE DISTANCE BETWEEN P AND Q?

8 LET R (–2, 4) AND T (5, 4) BE POINTS ON THE COORDINATE PLANE.

A PLOT THE POINTS R AND T.

B IS THE LINE THROUGH R AND T VERTICAL OR HORIZONTAL?

C WHAT IS THE DISTANCE BETWEEN POINTS R AND T?

Distance between points in a plane

SUPPOSE P (x_1, y_1) AND Q (x_2, y_2) ARE TWO DISTINCT POINTS ON THE xy-COORDINATE PLANE. WE CAN FIND THE DISTANCE BETWEEN THE TWO POINTS P AND Q BY CONSIDERING THREE CASES.

Case i WHEN P AND Q ARE ON A LINE PARALLEL TO THE x-AXIS (THAT IS, $\overline{PQ}$ IS A HORIZONTAL SEGMENT) AS IN FIGURE 4.2.

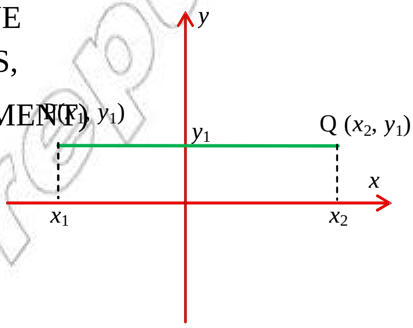


Figure 4.2

SINCE THE TWO POINTS P AND Q HAVE THE SAME y-COORDINATE (*ordinate*), THE DISTANCE BETWEEN P AND Q IS

$$PQ = |x_2 - x_1|$$

Case ii WHEN P AND Q ARE ON A LINE PARALLEL TO THE y-AXIS (THAT IS, $\overline{PQ}$ IS A VERTICAL SEGMENT) AS IN FIGURE 4.3.

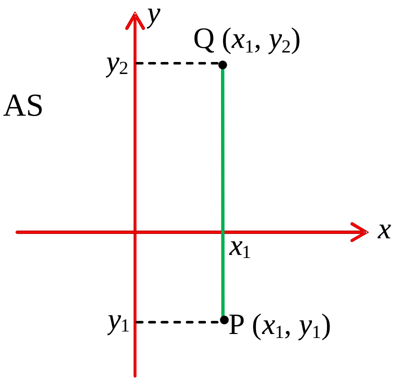


Figure 4.3

SINCE THE TWO POINTS HAVE THE SAME x-COORDINATE (*abscissa*), THE DISTANCE BETWEEN P AND Q IS

$$PQ = |y_2 - y_1|$$

Case iii WHEN $\overline{PQ}$ IS NEITHER VERTICAL NOR HORIZONTAL (THE GENERAL CASE).

TO FIND THE DISTANCE BETWEEN THE POINTS P AND Q, DRAW A LINE PASSING THROUGH P PARALLEL TO THE x-AXIS AND DRAW A LINE PASSING THROUGH Q PARALLEL TO THE y-AXIS.

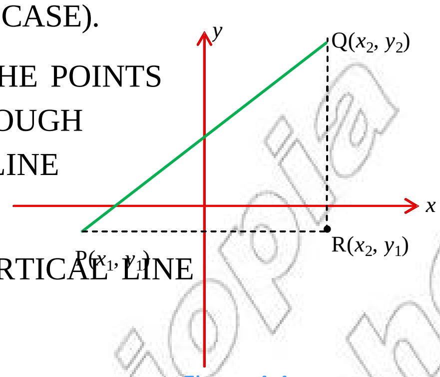


Figure 4.4

THE HORIZONTAL LINE AND THE VERTICAL LINE INTERSECT AT $R(x_2, y_1)$.

USING CASE i AND CASE ii, WE HAVE

$$PR = |x_2 - x_1| \text{ AND } RQ = |y_2 - y_1|$$

SINCE $\triangle PRQ$ IS A RIGHT ANGLED TRIANGLE, YOU CAN USE ***Pythagoras' Theorem*** TO FIND THE DISTANCE BETWEEN POINTS P AND Q AS FOLLOWS:

$$PQ^2 = PR^2 + RQ^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

THEREFORE, $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

THE RADICAL HAS POSITIVE SIGN (WHY?).

IN GENERAL, THE DISTANCE d BETWEEN ANY TWO POINTS $P(x_1, y_1)$ AND $Q(x_2, y_2)$ IS GIVEN BY

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

THIS IS CALLED THE ***distance formula***.

EXAMPLE 1 FIND THE DISTANCE BETWEEN THE GIVEN POINTS.

A A $(1, \sqrt{2})$ AND B $(1, -\sqrt{2})$ **B** P $\left(\frac{17}{4}, -2\right)$ AND Q $\left(\frac{1}{4}, -2\right)$

C R $\left(-\sqrt{2}, -1\right)$ AND S $\left(\sqrt{2}, -\sqrt{2}\right)$ **D** A $(a, -b)$ AND B $(-b, a)$

SOLUTION:

A $AB = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(1-1)^2 + (-\sqrt{2} - \sqrt{2})^2}$

$= \sqrt{(0)^2 + (-2\sqrt{2})^2} = 2\sqrt{2}.$

OR, MORE SIMPLY

$AB = |y_2 - y_1| = \left|-\sqrt{2} - \sqrt{2}\right|$

$= 2\sqrt{2}$ UNITS

B $PQ = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{\left(\frac{1}{4} - \frac{17}{4}\right)^2 + (-2-(-2))^2}$

$= \sqrt{\left(\frac{-16}{4}\right)^2 + (0)^2} = \sqrt{(-4)^2} = \sqrt{16} = 4$ UNIT

OR, MORE SIMPLY

$PQ = |x_2 - x_1| = \left|\frac{1}{4} - \frac{17}{4}\right|$

$= 4$ UNITS

C $RS = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left(\sqrt{2}-(-\sqrt{2})\right)^2 + \left(-\sqrt{2}-(-1)\right)^2}$

$= \sqrt{\left(2\sqrt{2}\right)^2 + \left(1-\sqrt{2}\right)^2} = \sqrt{11 - 2\sqrt{2}}$

D $AB = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-b-a)^2 + (a-(-b))^2}$

$= \sqrt{(b+a)^2 + (a+b)^2} = \sqrt{2(a+b)^2} = \sqrt{2}\,|a+b|$ UNIT

Exercise 4.1

1 IN EACH OF THE FOLLOWING, FIND THE DISTANCE BETWEEN THE TWO GIVEN POINT

A A (1, –5) AND B (7, 3)

B C $\left(-2, \frac{1}{2}\right)$ AND D $\left(\frac{1}{2}, 2\right)$

C E$(\sqrt{2}, 1)$ AND F$(-\sqrt{6}, \sqrt{3})$

D G $(a, -b)$ AND H$(-a, b)$

E THE ORIGIN AND K$\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$

F L$\left(\sqrt{2}, 1\right)$ AND M$\left(1, \sqrt{2}\right)$

G P$\left(\sqrt{2}, \sqrt{3}\right)$ AND Q$\left(2\sqrt{2}, 2\sqrt{3}\right)$

H R$\left(\sqrt{2}a, c\right)$ AND T$\left(\sqrt{2}b, c\right)$

2 USING THE DISTANCE FORMULA, SHOW THAT THE DISTANCE BETWEEN P AND Q IS:

A $|x_2 - x_1|$, WHEN $\overline{PQ}$ IS HORIZONTAL **B** $|y_2 - y_1|$, WHEN $\overline{PQ}$ IS VERTICAL.

3 LET A (3, –7) AND B (–1, 4) BE TWO ADJACENT VERTICES OF A SQUARE. CALCULATE T
OF THE SQUARE.

4 P (3, 5) AND Q (1, –3) ARE TWO OPPOSITE VERTICES OF A SQUARE. FIND ITS AREA.

5 SHOW THAT THE PLANE FIGURE WITH VERTICES:

A A (5, –1), B (2, 3) AND C (1, 1) IS A RIGHT ANGLED TRIANGLE.

B A (–4, 3), B (4, –3) AND C $(3\sqrt{3}, 4\sqrt{3})$ IS AN EQUILATERAL TRIANGLE.

C A (2, 3), B (6, 8), C (7, –1) IS AN ISOSCELES TRIANGLE.

6 AN EQUILATERAL TRIANGLE HAS TWO VERTICES AT A (–4, 0) AND B (4, 0). WHAT CO
COORDINATES OF THE THIRD VERTEX BE?

7 WHAT ARE THE POSSIBLE VALUES OF b IF THE POINT A(b, 4) IS 10 UNITS AWAY FROM B (0, –2)?

4.2 DIVISION OF A LINE SEGMENT

Recall that, a line segment passing through two points A and B is ***horizontal*** if the two points have the same ***y-coordinate***. I.e., a line segment whose end-points are A (x_1, y) and B (x_2, y) is a horizontal line segment as shown in Figure 4.5.

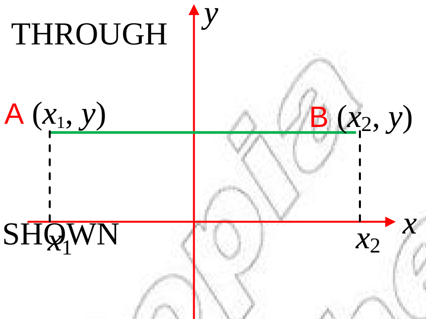
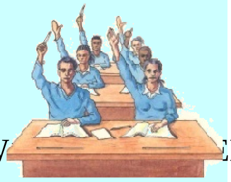


Figure 4.5

What is the mid-point of $\overline{AB}$?

ACTIVITY 4.2



1. Define the ratio of two quantities.
2. What is meant by the ratio of the length of two line segments?
3. In Figure 4.6, find the ratio of the length of $\overline{AP}$ to $\overline{PB}$.

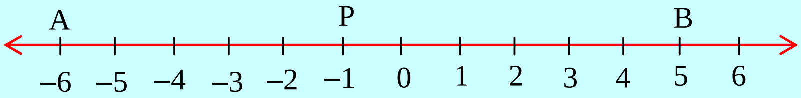


Figure 4.6

4. What is meant by a point P that divides a line segment AB internally?
5. Plot the following points on the coordinate plane and find the mid-point of the line segment joining the points.

   a A (2, –1) and B (2, 5) **b** C (–3, 3) and D (3, 3) **c** E (2, 0) and F (–2, 4).

Consider the horizontal line segment with end-points A (x_1, y) and B (x_2, y) as shown in Figure 4.7. In terms of the coordinates of A and B, determine the coordinates of the point P (x_o, y_o) that divides $\overline{AB}$ internally in the ratio $m:n$.

Clearly, the ratio of the line segment AP to the line segment PB is given by $\frac{AP}{PB}$.

The distance between A and P is $AP = x_o - x_1$.

The distance between P and B is $PB = x_2 - x_o$.

Therefore, $\frac{AP}{PB} = \frac{m}{n}$ i.e., $\frac{x_o - x_1}{x_2 - x_o} = \frac{m}{n}$.

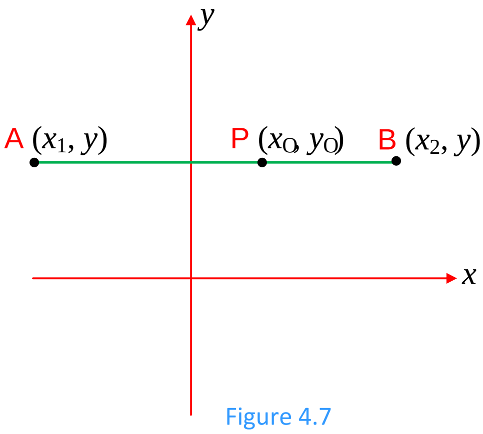


Figure 4.7

Solving this equation for x_o:

$\Rightarrow n(x_o - x_1) = m(x_2 - x_o)$

$\Rightarrow nx_o - nx_1 = mx_2 - mx_o$

$\Rightarrow nx_o + mx_o = nx_1 + mx_2$

$$\Rightarrow x_O(n+m) = nx_1 + mx_2$$

$$\Rightarrow x_O = \frac{nx_1+mx_2}{n+m}$$

SINCE $\overline{AB}$ IS PARALLEL TO THE x-AXIS (I.E. $\overline{AB}$ IS A HORIZONTAL LINE SEGMENT) AND OBVIOUSLY, $y_O = y$, THEREFORE, THE POINT $O(x_O, y_O)$ IS $\left(\frac{nx_1+mx_2}{n+m}, y\right)$.

GIVEN A LINE SEGMENT PQ WITH END POINT COORDINATES $P(x_1, y_1)$ AND $Q(x_2, y_2)$, LET US FIND THE COORDINATES OF THE POINT R DIVIDING THE LINE SEGMENT PQ INTERNALLY IN THE RATIO $m : n$ I.E, $\frac{PR}{RQ} = \frac{m}{n}$, WHERE m AND n ARE GIVEN POSITIVE REAL NUMBERS.

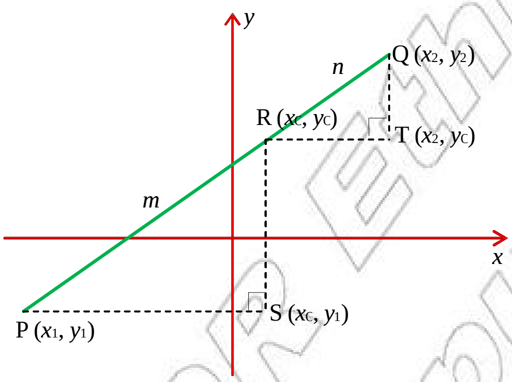


Figure 4.8

LET THE COORDINATES OF R BE (x_O, y_O). ASSUME THAT $x_1 \neq x_2$ AND $y_1 \neq y_2$.

IF YOU DRAW LINES THROUGH THE POINTS P, R AND Q PARALLEL TO THE AXES AS SHOWN IN FIGURE 4.8, THE POINTS S AND T HAVE THE COORDINATES (x_O, y_1) AND (x_2, y_O), RESPECTIVELY.

$PS = x_O - x_1$, $RT = x_2 - x_O$, $SR = y_O - y_1$ AND $TQ = y_2 - y_O$

SINCE TRIANGLES PSR AND RTQ ARE SIMILAR (WHY?),

$$\frac{PS}{RT} = \frac{PR}{RQ} \text{ AND } \frac{SR}{TQ} = \frac{PR}{RQ}$$

$$\frac{x_o - x_1}{x_2 - x_o} = \frac{m}{n} \text{ AND } \frac{y_o - y_1}{y_2 - y_o} = \frac{m}{n}$$

SOLVING FOR x_O AND y_O

$\Rightarrow \quad n(x_O - x_1) = m(x_2 - x_O)$ AND $n(y_O - y_1) = m(y_2 - y_O)$

$\Rightarrow \quad nx_O - nx_1 = mx_2 - mx_O$ AND $ny_O - ny_1 = my_2 - my_O$

$\Rightarrow \quad nx_O + mx_O = nx_1 + mx_2$ AND $ny_O + my_O = ny_1 + my_2$

$\Rightarrow \quad x_O(n+m) = nx_1 + mx_2$ AND $y_O(n+m) = ny_1 + my_2$

$$\Rightarrow \quad x_O = \frac{nx_1+mx_2}{n+m} \text{ AND } y_o = \frac{ny_1+my_2}{n+m}$$

THE POINT $R(x_O, y_O)$ DIVIDING THE LINE SEGMENT PQ INTERNALLY IN THE RATIO $m : n$ IS GIVEN BY

$$R(x_O, y_O) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right)$$

THIS IS CALLED THE ***section formula***.

EXAMPLE 1 FIND THE COORDINATES OF THE POINT R THAT DIVIDES THE LINE SEGMENT WITH END- POINTS A (6, 2) AND B (1, – 4) IN THE RATIO 2:3.

SOLUTION: PUT $(x_1, y_1) = (6, 2)$, $(x_2, y_2) = (1, -4)$, $m = 2$ AND $n = 3$. USING THE SECTION FORMULA, YOU HAVE

$$R(x_O, y_O) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right) = \left(\frac{3\times 6 + 2\times 1}{3+2}, \frac{3\times 2 + 2\times(-4)}{3+2}\right)$$

$$= \left(\frac{18+2}{5}, \frac{6-8}{5}\right) = \left(4, -\frac{2}{5}\right)$$

THEREFORE, R IS $\left(4, -\frac{2}{5}\right)$.

EXAMPLE 2 A LINE SEGMENT HAS END-POINTS (–2, –3) AND (7, 12) AND IT IS DIVIDED INT THREE EQUAL PARTS. FIND THE COORDINATES OF THE POINTS THAT TRISECT SEGMENT.

SOLUTION: THE FIRST POINT DIVIDES THE LINE SEGMENT IN THE RATIO 1:2, AND HENCE

$$x_O = \frac{nx_1 + mx_2}{n+m} \text{ AND } y_o = \frac{ny_1 + my_2}{n+m}$$

SO, $x_O = \dfrac{2\times(-2) + 1\times 7}{1+2}$ AND $y_O = \dfrac{2\times(-3) + 1\times 12}{1+2}$

$$\Rightarrow x_O = \frac{-4+7}{3} \text{ AND } y_O = \frac{-6+12}{3} \Rightarrow x_O = 1 \text{ AND } y_O =$$

THEREFORE, THE FIRST POINT IS (1, 2).

THE SECOND POINT DIVIDES THE LINE SEGMENT IN THE RATIO 2:1. THUS,

$$x_O = \frac{nx_1 + mx_2}{n+m} \text{ AND } y_o = \frac{ny_1 + my_2}{n+m}$$

SO, $x_o = \dfrac{1\times(-2) + 2\times 7}{1+2}$ AND $y_O = \dfrac{1\times(-3) + 2\times 12}{1+2}$

$$\Rightarrow x_O = \frac{-2+14}{3} \text{ AND } y_O = \frac{-3+24}{3}$$

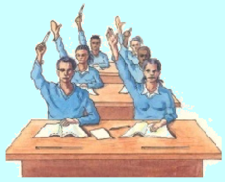
$\Rightarrow x_O = 4$ AND $y_O = 7$.

THEREFORE, THE SECOND POINT IS (4, 7).

The mid-point formula

A POINT THAT DIVIDES A LINE SEGMENT INTO TWO EQUAL PARTS IS THE MID-POINT OF T

ACTIVITY 4.3



1 CONSIDER THE POINTS P (2, 1) AND Q (12, 1).

A FIND THE DISTANCE BETWEEN *P* AND *Q*.

B IF R IS A POINT WITH COORDINATES (7, 1),

I FIND *PR*. **II** FIND *RQ*

III IS *PR* EQUAL TO *RQ*? **IV** WHAT IS THE MID-POINT OF *PQ*?

C DIVIDE $\overline{PQ}$ IN THE RATIO 1:1.

2 FIND THE COORDINATES OF THE MID-POINT OF EACH OF THE FOLLOWING LINE SEG END-POINTS:

I P (x_1, y_1) AND Q (x_1, y_2). **II** R (x_1, y_1) AND S (x_2, y_1).

WHICH OF THE ABOVE SEGMENTS ARE HORIZONTAL?

LET P (x_1, y_1) AND Q (x_2, y_2) BE THE END-POINTS OF $\overline{PQ}$.

IF *PR* = *RQ* (THE CASE WHERE $m = n$), THEN R IS THE MID-POINT OF THE LINE SEGMENT *PQ*

NOW LET US DERIVE THE MID-POINT FORMULA.

$$R\,(x_0, y_0) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right)$$

$$= \left(\frac{nx_1 + nx_2}{n+n}, \frac{ny_1 + ny_2}{n+n}\right) = \left(\frac{n\,(x_1 + x_2)}{2n}, \frac{n\,(y_1 + y_2)}{2n}\right) (\text{AS } m = n)$$

$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

THIS IS THE FORMULA USED TO FIND THE ***mid-point*** OF THE LINE SEGMENT *PQ* WHOSE END POINTS ARE P (x_1, y_1) AND Q (x_2, y_2).

THE ***mid-point*** OF THE LINE SEGMENT JOINING THE POINTS (x_1, y_1) AND (x_2, y_2) IS GIVEN BY

$$M\,(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

EXAMPLE 3 FIND THE COORDINATES OF THE MID-POINT OF THE LINE SEGMENTS WITH END-POINTS:

A P (–3, 2) AND Q (5, –4)

B P $(3-\sqrt{2}, 3+\sqrt{2})$ AND Q $(1+\sqrt{2}, 3-\sqrt{2})$.

SOLUTION:

A $M(x_O, y_O) = \left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

$x_O = \dfrac{x_1+x_2}{2}$ AND $y_o = \dfrac{y_1+y_2}{2}$

$x_O = \dfrac{-3+5}{2} = 1$ AND $y_O = \dfrac{2-4}{2} = -$

THEREFORE $M(x_O, y_O) = (1, -1)$.

B $M(x_O, y_O) = \left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

$x_O = \dfrac{x_1+x_2}{2}$ AND $y_o = \dfrac{y_1+y_2}{2}$

$x_O = \dfrac{3-\sqrt{2}+1+\sqrt{2}}{2}$ AND $y_O = \dfrac{3+\sqrt{2}+3-\sqrt{2}}{2}$

$x_O = \dfrac{4}{2} = 2$ AND $y_O = \dfrac{6}{2} = 3$

THEREFORE $M(x_O, y_O) = (2, 3)$.

Group Work 4.1

1 A LINE SEGMENT HAS END-POINTS P (–3, 1) AND Q (5

A WHAT IS THE LENGTH OF THE LINE SEGMENT?

B FIND THE COORDINATES OF THE MID-POINT OF THE SEGMENT.

2 A LINE SEGMENT HAS ONE END-POINT AT A (4, 3). IF ITS MID-POINT IS AT M (1, –1), WHERE IS THE OTHER END-POINT?

3 FIND THE POINTS THAT DIVIDE THE LINE SEGMENT WITH END-POINTS AT P (4, –3) AND Q (–6, 7) INTO THREE EQUAL PARTS.

4 LET A (–2, –1), B (6, –1), C (6, 3) AND D (–2, 3) BE VERTICES OF A RECTANGLE. SUPPOSE *P*, *Q*, *R* AND *S* ARE MID-POINTS OF THE SIDES OF THE RECTANGLE.

I WHAT IS THE AREA OF RECTANGLE *ABCD*?

II WHAT IS THE AREA OF QUADRILATERAL *PQRS*?

III GIVE THE RATIO OF THE AREAS IN I AND II.

Exercise 4.2

1. FIND THE COORDINATES OF THE MID-POINT OF THE LINE SEGMENTS JOINING THE POINTS
   - **A** A (1, 4) AND B (−2, 2)
   - **B** (*a*, *b*) AND THE ORIGIN
   - **C** M (*p*, *q*) AND N (*q*, *p*)
   - **D** $\left(1\frac{1}{2}, -1\right)$ AND $\left(-\frac{5}{2}, 1\right)$
   - **E** E$(1+\sqrt{2}, \sqrt{2})$ AND F$(2-\sqrt{2}, \sqrt{8})$
   - **F** G$\left(\sqrt{5}, 1-\sqrt{3}\right)$ AND H$\left(3\sqrt{5}, 1+\sqrt{3}\right)$
2. THE MID-POINT OF A LINE SEGMENT IS M (−3, 2). ONE END-POINT OF THE SEGMENT IS P (1, −3). FIND THE COORDINATES OF THE OTHER END-POINT.
3. FIND THE COORDINATES OF THE POINT *R* THAT DIVIDES THE LINE SEGMENT JOINING THE POINTS A (1, 3) AND B (−4, −3) IN THE RATIO 2:3.
4. A LINE SEGMENT HAS END-POINTS P (−1, 5) AND Q (5, 2). FIND THE COORDINATES OF THE POINTS THAT TRISECT THE SEGMENT.
5. FIND THE MID-POINTS OF THE SIDES OF THE TRIANGLE WITH VERTICES A (−1, 3), B (4, 6) AND C (3, −1).

4.3 EQUATION OF A LINE

4.3.1 Gradient (slope) of a Line

FROM YOUR EVERYDAY EXPERIENCE, YOU MIGHT BE FAMILIAR WITH THE IDEA OF GRADIENT (SLOPE).

A ***hill*** MAY BE ***steep*** OR MAY RISE VERY SLOWLY. THE NUMBER THAT DESCRIBES THE ***steepness*** OF A HILL IS CALLED THE ***gradient (slope)*** OF THE HILL

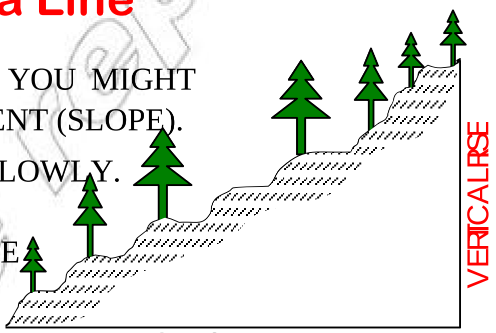
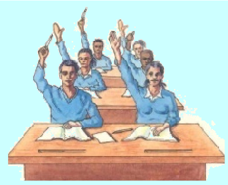


Figure 4.9

WE MEASURE THE GRADIENT OF A HILL BY THE RATIO OF THE ***vertical rise*** TO THE ***horizontal run.***

ACTIVITY 4.4



GIVEN POINTS P (1, 2), Q (−1, −4), R (0, –1) AND S (3, 8)

A FIND THE VALUE OF $\frac{y_2 - y_1}{x_2 - x_1}$ TAKING

I *P* AND *Q* **II** *P* AND *R* **III** *Q* AND *R* **IV** *R* AND *S*

B ARE THE VALUES OBTAINED IN A ABOVE EQUAL? WHAT DO YOU CALL THESE VALUES?

IN COORDINATE GEOMETRY, THE GRADIENT OF A NON-VERTICAL STRAIGHT LINE IS THE RATIO OF "CHANGE IN y-COORDINATES" TO THE CORRESPONDING "CHANGE IN x-COORDINATES". THAT IS, THE SLOPE OF A LINE THROUGH P AND Q IS THE RATIO OF THE VERTICAL DISTANCE FROM R TO Q TO THE HORIZONTAL DISTANCE FROM P TO Q

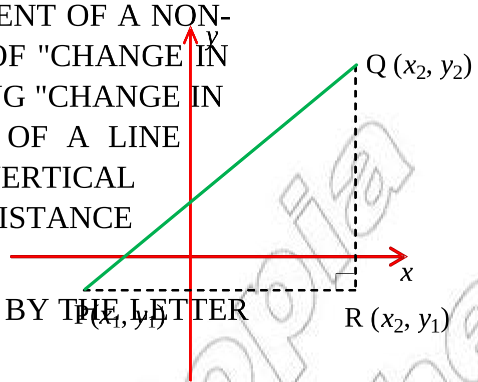


Figure 4.10

IF WE DENOTE THE GRADIENT OF A LINE BY THE LETTER m, THEN

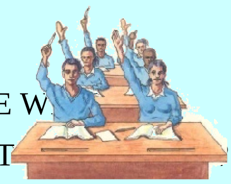
$$m = \frac{\text{change in } y\text{-coordinates}}{\text{change in } x\text{-coordinates}} = \frac{y_2 - y_1}{x_2 - x_1};\ x_1 \neq x_2$$

Definition 4.1

If (x_1, y_1) and (x_2, y_2) are points on a line with $x_1 \neq x_2$, then the ***gradient*** of the line, denoted by m, is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

ACTIVITY 4.5



1 IF A (x_1, y_1) AND B (x_2, y_2) ARE DISTINCT POINTS ON A LINE WITH $x_1 = x_2$, THEN WHAT CAN BE SAID ABOUT THE GRADIENT? IS THE LINE VERTICAL OR HORIZONTAL?

2 WHAT IS THE GRADIENT OF ANY HORIZONTAL LINE?

3 CONSIDER THE LINE WITH EQUATION $f(x) = 3x - 1$. TAKE THREE DISTINCT POINTS P_1, P_2 AND P_3 ON THE LINE.

A FIND THE GRADIENT USING P_1 AND P_2 **B** FIND THE GRADIENT USING P_1 AND P_3.

C WHAT DO YOU OBSERVE FROM A AND B?

4 LET P_1, P_2, P_3 AND P_4 BE POINTS ON A NON-VERTICAL STRAIGHT LINE $y = ax + b$ WITH COORDINATES (x_1, y_1), (x_2, y_2), (x_3, y_3) AND (x_4, y_4) RESPECTIVELY. FIND:

A THE GRADIENT OF THE LINE TAKING P_1 AND P_2

B THE GRADIENT OF THE LINE TAKING P_3 AND P_4

C ARE THE RATIOS $\frac{y_2 - y_1}{x_2 - x_1}$ AND $\frac{y_4 - y_3}{x_4 - x_3}$ EQUAL?

D COULD YOU CONCLUDE THAT THE GRADIENT OF A LINE DOES NOT DEPEND ON OF POINTS ON THE LINE?

EXAMPLE 1 FIND THE GRADIENT OF THE LINE PASSING THROUGH EACH OF THE FOLLOWING PAIRS OF POINTS:

A P (–7, 2) AND Q (4, 3) **B** A $\left(\sqrt{2}, 1\right)$ AND B $\left(-\sqrt{2}, -3\right)$

C P (2, –3) AND Q (5, –3) **D** A $\left(-\dfrac{1}{2}, -2\right)$ AND B $\left(-\dfrac{1}{2}, 2\right)$.

SOLUTION:

A $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-2}{4-(-7)} = \dfrac{1}{11}$

B $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-3-1}{-\sqrt{2}-\sqrt{2}} = \dfrac{-4}{-2\sqrt{2}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

C $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-3-(-3)}{5-2} = \dfrac{-3+3}{3} = \dfrac{0}{3} = 0$

SO, $m = 0$. IS THE LINE HORIZONTAL? WHAT IS ITS EQUATION?

D $x_1 = -\dfrac{1}{2}$ AND $x_2 = -\dfrac{1}{2}$

THE LINE IS VERTICAL. SO IT HAS NO MEASURABLE GRADIENT.

THE EQUATION OF THE LINE IS $x = x_1 = x_2 = -\dfrac{1}{2}$ OR SIMPLY $x = -\dfrac{1}{2}$

Note: GRADIENT FOR A VERTICAL LINE IS NOT DEFINED.

EXAMPLE 2 CHECK THAT THE LINE ℓ_1 THROUGH P (0, 1) AND Q (–1, 4) AND ℓ_2 THROUGH R $\left(\dfrac{2}{3}, 0\right)$ AND T (1, –1) HAVE SAME GRADIENTS. ARE THE LINES PARALLEL?

SOLUTION: FOR ℓ_1, $m_1 = \dfrac{4-1}{-1-0} = \dfrac{3}{-1} = -3$. FOR ℓ_2, $m_2 = \dfrac{-1-0}{1-\dfrac{2}{3}} = \dfrac{-1}{\dfrac{1}{3}} = -3$.

HERE, $m_1 = m_2$. DRAW THE LINES AND SEE THAT ℓ_1 IS PARALLEL TO ℓ_2.

Exercise 4.3

1 FIND THE GRADIENTS OF THE LINES PASSING THROUGH THE FOLLOWING POINTS:

A A (4 , 3) AND B (8, 11) **B** P (3, 7) AND Q (1, 9)

C C $(\sqrt{2}, -9)$ AND D $(2\sqrt{2}, -7)$ **D** R (–5, –2) AND S (7, –8)

E E (5, 8) AND F (–2, 8) **F** H (1, 7) AND K (1, –6)

G R (1, b) AND S (b, a) , $b \neq 1$.

2 A (2, −3), B (7, 5) AND C (−2, 9) ARE THE VERTICES OF TRIANGLE ABC. FIND THE GRADIENT OF EACH OF THE SIDES OF THE TRIANGLE.

3 GIVEN THREE POINTS P (−1, −5), Q (1, −2) AND R (5, 4), FIND THE GRADIENT OF $\overline{PQ}$ AND $\overline{QR}$. WHAT DO YOU CONCLUDE FROM YOUR RESULT?

4 USE GRADIENTS TO SHOW THAT THE POINTS $P_1(-4, 6)$, $P_2(-1, 12)$ AND $P_3(-7, 0)$ ARE COLLINEAR, I.E., ALL LIE ON THE SAME STRAIGHT LINE.

5 SHOW THAT THE LINE PASSING THROUGH THE POINTS $A\left(0, \frac{3}{2}\right)$ AND B(−2, 3), ALSO PASSES THROUGH THE POINT C(−6, −9).

4.3.2 Slope of a Line in Terms of Angle of Inclination

THE ANGLE MEASURED FROM THE POSITIVE x-AXIS TO A LINE, IN ANTICLOCKWISE DIRECTION, IS CALLED THE ***inclination of the line*** OR THE ANGLE OF INCLINATION OF THE LINE. THIS ANGLE IS ALWAYS LESS THAN 180°.

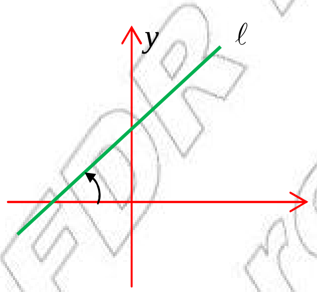


Figure 4.11

Group Work 4.2

CONSIDER THE RIGHT ANGLED TRIANGLE OAB IN FIGURE 4.12.

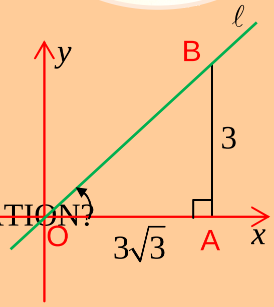


Figure 4.12

1 HOW LONG IS THE HYPOTENUSE $\overline{OB}$?

2 WHAT IS TANGENT OF ANGLE BOA?

3 WHAT IS MEASURE OF ANGLE BOA?

4 WHAT IS THE ANGLE OF INCLINATION OF LINE ℓ?

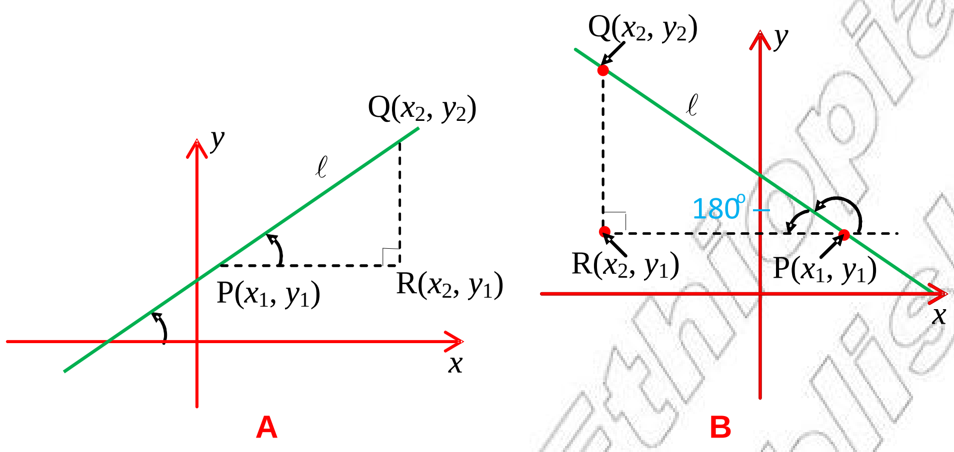
5 WHAT IS THE TANGENT OF THE ANGLE OF INCLINATION?

6 BY FINDING THE COORDINATES OF B, CALCULATE THE SLOPE OF LINE ℓ.

7 WHAT RELATIONSHIP DO YOU SEE BETWEEN YOUR ANSWERS FOR QUESTIONS 5 AND 6 ABOVE?

THE ABOVE GROUP WORK WILL HELP YOU TO UNDERSTAND THE RELATIONSHIP BETWEEN SLOPE AND ANGLE OF INCLINATION.

FOR A NON-VERTICAL LINE, THE ***tangent*** OF THIS ANGLE IS THE ***slope*** OF THE LINE. OBSERVE THE FOLLOWING.



A B

Figure 4.13

IN FIGURE 4.13A ABOVE, AS $y_2 - y_1$ REPRESENTS THE DISTANCE RQ AND $x_2 - x_1$ REPRESENTS THE DISTANCE PR, THE SLOPE OF THE STRAIGHT LINE ℓ IS ACTUALLY REPRESENTED BY THE RATIO

$$m = \frac{RQ}{PR} = \frac{y_2 - y_1}{x_2 - x_1} = \tan(m\angle RPQ)$$

$$\therefore m = \tan\theta$$

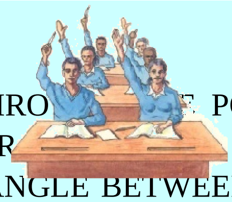
A LINE MAKING AN ACUTE ANGLE OF INCLINATION θ WITH THE POSITIVE DIRECTION OF THE x-AXIS HAS POSITIVE SLOPE.

SIMILARLY, A LINE WITH OBTUSE ANGLE OF INCLINATION, (SEE FIGURE 4.13B), HAS NEGATIVE SLOPE.

$$\text{SLOPE OF } \ell = \frac{RQ}{PR} = \frac{y_2 - y_1}{x_1 - x_2} = -\frac{y_2 - y_1}{x_2 - x_1} = -\tan(180° - \theta) = -(-\tan\theta) = \tan\theta$$

(*In Unit 5, this will be clarified*)

ACTIVITY 4.6



1. HOW WOULD YOU DESCRIBE THE LINE PASSING THRO POINTS WITH COORDINATES (x_1, y_1) AND (x_1, y_2)? IS IT PERPENDICULAR x-AXIS OR THE y-AXIS? WHAT IS THE TANGENT OF THE ANGLE BETWEEN THIS LINE AND THE x-AXIS?
2. SUPPOSE A LINE PASSES THROUGH THE POINTS WITH COORDINATES (x_1, y_1) AND (x_2, y_1). FIND THE TANGENT OF THE ANGLE FORMED BY THIS LINE AND THE x-AXIS. WHAT IS THE SLOPE OF THIS LINE?
3. WHAT IS THE ANGLE OF INCLINATION OF THE LINE $y = x$? AND THE LINE $y = -x$?

IN GENERAL, THE SLOPE OF A LINE MAY BE EXPRESSED IN TERMS OF THE COORDINATES (x_1, y_1) AND (x_2, y_2) ON THE LINE AS FOLLOWS:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta, \; x_2 \neq x_1$$

WHERE θ IS THE ANTICLOCKWISE ANGLE BETWEEN THE POSITIVE x-AXIS AND THE LINE.

EXAMPLE 3 FIND THE SLOPE OF A LINE, IF ITS INCLINATION IS:

A 60° **B** 135°

SOLUTION:

A SLOPE : $m = \tan\theta = \tan 60^{\circ} = \sqrt{3}$

B SLOPE : $m = \tan\theta = \tan 135^{\circ} = \tan(180^{\circ} - 45^{\circ}) = -\tan 45^{\circ} = -1$

Note: IF θ IS AN OBTUSE ANGLE, THEN $\tan\theta = -\tan(180^{\circ} - \theta)$.

EXAMPLE 4 FIND THE ANGLE OF INCLINATION OF THE LINE

A CONTAINING THE POINTS A(3, –3) AND B(–1, 1)

B CONTAINING THE POINTS C(0, 5) AND D(4, 5).

SOLUTION:

A $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-3)}{-1-3} = -1$. SO $\tan\theta = -1$ AND HENCE $\theta = 135^{\circ}$.

B $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-5}{4-0} = 0$, $\tan\theta = 0$. SO, $\theta = 0^{\circ}$.

Note: LET m BE THE SLOPE OF A NON-VERTICAL LINE.

I IF $m > 0$, THEN THE LINE RISES FROM LEFT TO RIGHT AS SHOWN IN FIGURE 4.14A.

II IF $m < 0$, THEN THE LINE FALLS FROM LEFT TO RIGHT AS SHOWN IN FIGURE 4.14B.

III IF $m = 0$, THEN THE LINE IS HORIZONTAL AS IN FIGURE 4.14C.

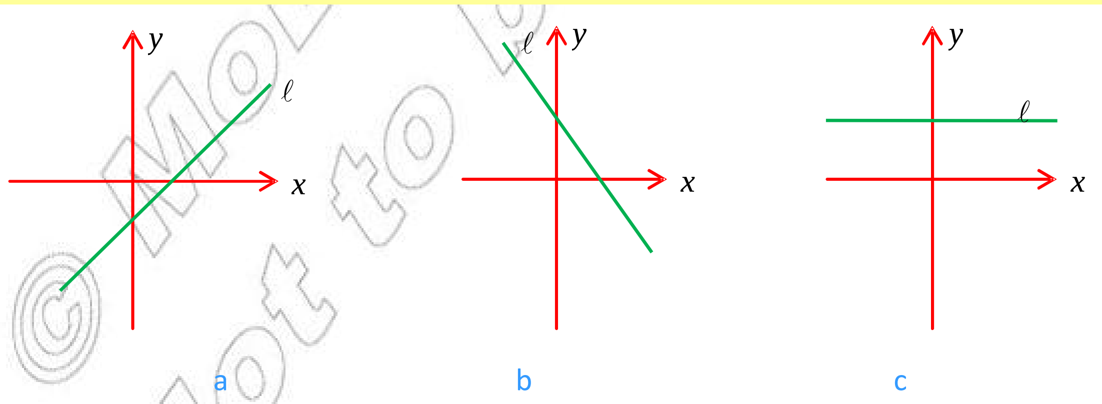


Figure 4.14

Exercise 4.4

1 FIND THE SLOPE OF THE LINE WHOSE ANGLE OF INCLINATION IS:

A 30^{O} **B** 75^{O} **C** 150^{O} **D** 90^{O} **E** 0^{O}.

2 FIND THE ANGLE OF INCLINATION OF THE LINE IF ITS SLOPE IS:

A $-\sqrt{3}$ **B** $\dfrac{-\sqrt{3}}{3}$ **C** 1 **D** $\dfrac{1}{\sqrt{3}}$ **E** 0.

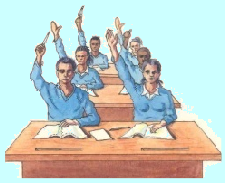
3 THE POINTS A(–2, 0), B (0, 2) AND C (2, 0) ARE VERTICES OF A TRIANGLE. FIND THE MEASURE OF THE THREE ANGLES OF TRIANGLE ABC. WHAT TYPE OF TRIANGLE IS IT?

4.3.3 Different Forms of Equations of a Line

FROM EUCLIDEAN GEOMETRY, YOU MAY RECALL THAT THERE IS A UNIQUE LINE PAS TWO DISTINCT POINTS. THE EQUATION OF A LINE IS AN EQUATION IN x AND y WHICH IS SATISFIED BY THE COORDINATES OF EVERY POINT ON THE LINE AND IS NOT SATISFIED BY THE COORDINATES OF ANY POINT NOT ON THE LINE.

THE EQUATION OF A STRAIGHT LINE CAN BE EXPRESSED IN DIFFERENT FORMS. SOME O THE POINT-SLOPE FORM, THE SLOPE-INTERCEPT FORM AND THE TWO-POINT FORM.

ACTIVITY 4.7



1 SHOW THAT THE GRAPH OF THE EQUATION $x = 2$ CONTAINS POINTS A (2, 0), B (2, –1), C (2, 2) AND D$\left(2, \dfrac{1}{3}\right)$.

2 CONSIDER THE GRAPH OF THE STRAIGHT LINE $y = x + 1$. DETERMINE WHICH OF THE FOLLOWING POINTS LIE ON THE LINE.

A (3, –1), B (–1, 0), C$\left(\dfrac{-1}{2}, \dfrac{3}{2}\right)$, D (0, 1), E$\left(\dfrac{-1}{2}, 1\right)$, F (–2, –1) AND G(–1, –2)

3 WHICH OF THE FOLLOWING POINTS LIE ON THE LINE $y = -5x + 4$?

A (–1, 9), B (–2, 12), C(0, 4), D$\left(\dfrac{2}{5}, 2\right)$, E (3, –10).

4 WHAT DO YOU CALL THE NUMBER b IF A LINE INTERSECTS THE y-AXIS AT POINT P (0, b)?

5 CONSIDER THE GRAPH OF THE STRAIGHT LINE $y = mx + b$. FIND ITS y-INTERCEPT AND x-INTERCEPT.

6 GIVE THE EQUATIONS OF THE LINES THROUGH THE POINTS:

A P (–1, 3) AND Q (4, 3) **B** R (–1, 1) AND S (1, –1).

The point-slope form of equation of a line

WE NORMALLY USE THIS FORM OF THE EQUATION OF A LINE IF THE SLOPE m OF THE LINE AND THE COORDINATES OF A POINT ON IT ARE GIVEN.

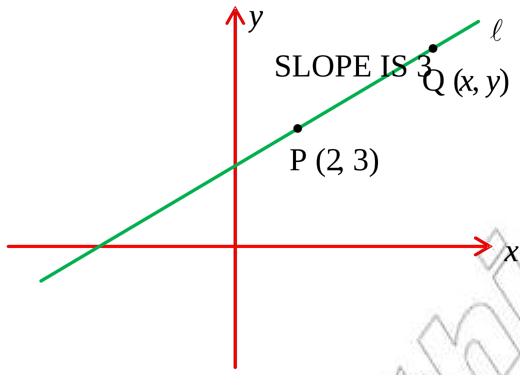


Figure 4.15

SUPPOSE YOU ARE ASKED TO FIND THE EQUATION OF THE STRAIGHT LINE WITH SLOPE 3 THROUGH THE POINT WITH COORDINATE (2, 3).

TAKE P TO BE THE POINT (2, 3) AND LET $Q(x, y)$ BE ANY OTHER POINT ON THE LINE AS SHOWN IN FIGURE 4.15. WHAT IS THE SLOPE OF THE STRAIGHT LINE JOINING THE POINTS WITH COORD (x_1, y_1) AND (x_2, y_2)?

WHAT IS THE SLOPE OF $\overleftrightarrow{PQ}$? YOU ARE GIVEN THAT THE SLOPE OF THIS LINE IS 3. IF YOU HA ANSWERED CORRECTLY, YOU SHOULD OBTAIN

$$y = 3x - 3;$$

WHICH IS THE REQUIRED EQUATION OF THE STRAIGHT LINE.

IN GENERAL, SUPPOSE YOU WANT TO FIND THE EQUATION OF THE STRAIGHT LINE THROUGH THE POINT WITH COORDINATES (x_1, y_1) AND WHICH HAS SLOPE m. AGAIN, LET THE POINT WITH GIVEN COORDINATES (x_1, y_1) BE A. TAKE ANY OTHER POINT ON THE LINE, SAY B, WITH COORDINATES (x, y) AS SHOWN IN FIGURE 4.16.

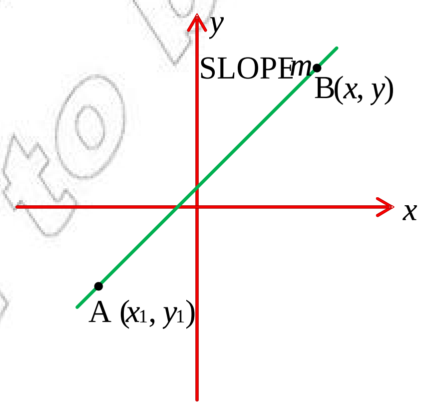


Figure 4.16

THEN THE SLOPE OF $\overleftrightarrow{AB}$ IS $\dfrac{y-y_1}{x-x_1}$

$\Rightarrow y - y_1 = m\,(x - x_1)$ WHICH IS THE SAME AS $y = y_1 + m\,(x - x_1)$.

THIS EQUATION IS CALLED THE ***point-slope form*** of the equation of a line

EXAMPLE 5 FIND THE EQUATION OF THE STRAIGHT LINE WITH SLOPE $\dfrac{-3}{2}$ AND WHICH PASSES THROUGH THE POINT (–3, 2).

SOLUTION: ASSUME THAT THE POINT (x, y) IS ANY POINT ON THE LINE OTHER THAN (–3, 2). THUS, USING THE EQUATION $y - y_1 = m\,(x - x_1)$

$$\Rightarrow\ y - 2 = \frac{-3}{2}\,(x + 3)$$

$$\Rightarrow\ y = -\frac{3}{2}x - \frac{5}{2}\ \text{ OR } 2y + 3x + 5 = 0.$$

The slope-intercept form of equation of a line

CONSIDER THE EQUATION $y = mx + b$. WHEN $x = 0$, $y = b$. ALSO, WHEN $x = 1$, $y = m + b$ AS SHOWN IN FIGURE 4.17.

YOU CAN SEE THAT P $(0, b)$ IS THE POINT WHERE THE LINE WITH EQUATION $y = mx + b$ CROSSES THE y-AXIS. (b IS CALLED THE ***y-intercept*** OF THE LINE). LET Q BE $(1, m + b)$.

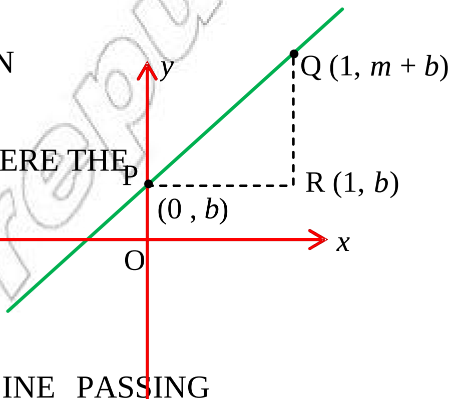


Figure 4.17

USING THE COORDINATES OF P AND Q, SHOW THAT THE SLOPE OF THE STRAIGHT LINE PASSING THROUGH P AND Q IS m

WRITING THE EQUATION OF THIS LINE THROUGH THE POINT $(0, b)$ WITH SLOPE m, USING THE POINT-SLOPE FORM, GIVES

$$y - b = m\,(x - 0) \Rightarrow y = mx + b$$

WHERE m IS SLOPE OF THE LINE AND b IS y-INTERCEPT OF THE LINE.

THIS EQUATION IS CALLED THE ***slope-intercept form*** OF THE EQUATION OF A LINE.

Note: THE SLOPE-INTERCEPT FORM OF EQUATION OF A LINE ENABLES US TO FIND THE THE y-INTERCEPT, ONCE THE EQUATION IS GIVEN.

EXAMPLE 6 FIND THE EQUATION OF THE LINE WITH SLOPE $-\frac{2}{3}$ AND y-INTERCEPT 3.

SOLUTION: HERE, $m = \frac{-2}{3}$ AND THE y-INTERCEPT IS 3.

THEREFORE, THE EQUATION OF THE LINE IS $y = -\frac{2}{3}x + 3$.

The two-point form of equation of a line

FINALLY, LET US LOOK AT THE SITUATION WHERE THE SLOPE OF A NON-VERTICAL LINE IS NOT GIVEN, BUT TWO POINTS ON THE LINE ARE GIVEN.

CONSIDER A STRAIGHT LINE WHICH PASSES THROUGH THE POINTS P(x_1, y_1) AND Q(x_2, y_2). IF R (x, y) IS ANY POINT ON THE LINE OTHER THAN P(x_1, y_1) OR Q(x_2, y_2), THEN THE SLOPE OF $\overleftrightarrow{PR}$ IS

$$m = \frac{y - y_1}{x - x_1},\ x \neq x_1$$

AND THE SLOPE OF $\overleftrightarrow{PQ}$ IS

$$m = \frac{y_2 - y_1}{x_2 - x_1},\ x_1 \neq x_2$$

BUT THE SLOPE OF $\overleftrightarrow{PR}$ = THE SLOPE OF $\overleftrightarrow{PQ}$

$$\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

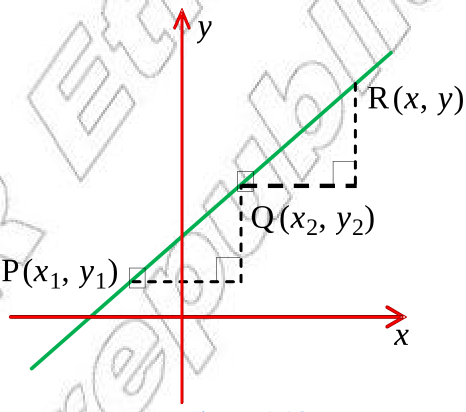


Figure 4.18

THIS EQUATION IS CALLED THE ***two-point form*** of the equation of a line.

EXAMPLE 7 FIND THE EQUATION OF THE LINE PASSING THROUGH THE POINTS P (–1, 5) AND Q (3, 13).

SOLUTION: TAKING (–1, 5) AS (x_1, y_1) AND (3, 13) AS (x_2, y_2), USE THE TWO-POINT FORM TO GET THE EQUATION OF THE LINE TO BE

$$y - 5 = \frac{13 - 5}{3 + 1}(x + 1) = 2x + 2 \text{ WHICH IMPLIES } y = 2x + 7$$

The general equation of a line

A FIRST DEGREE (LINEAR) EQUATION IN x AND y IS AN EQUATION OF THE FORM;

$$Ax + By + C = 0$$

WHERE A, B AND C ARE FIXED REAL NUMBERS SUCH THAT $A \neq 0$ OR $B \neq 0$.

ALL THE DIFFERENT FORMS OF EQUATIONS OF LINES DISCUSSED ABOVE CAN BE EXPRESS

$$Ax + By + C = 0$$

CONVERSELY, ONE CAN SHOW THAT ANY LINEAR EQUATION IN x AND y IS THE EQUATION OF A LINE. SUPPOSE A LINEAR EQUATION IN x AND y IS GIVEN AS

$$Ax + By + C = 0.$$

IF $B \neq 0$, THEN THE EQUATION MAY BE SOLVED FOR y AS FOLLOWS:

$$Ax + By + C = 0$$

$$By = -Ax - C$$

$$y = \frac{-A}{B}x - \frac{C}{B}$$

THIS EQUATION IS OF THE FORM $y = mx + b$ AND THEREFORE REPRESENTS A STRAIGHT LINE WITH SLOPE $m = -\frac{A}{B}$ AND y-INTERCEPT $b = -\frac{C}{B}$.

WHAT WILL BE THE EQUATION $Ax + By + C = 0$, IF $B = 0$ AND $A \neq 0$?

EXAMPLE 8 FIND THE SLOPE AND y-INTERCEPT OF THE LINE WHOSE GENERAL EQUATION IS $3x - 6y - 4 = 0$.

SOLUTION: SOLVING FOR y THE EQUATION $3x - 6y - 4 = 0$ GIVES,

$$-6y = -3x + 4 \Rightarrow y = \frac{-3x}{-6} + \frac{4}{-6} = \frac{1}{2}x - \frac{2}{3}$$

SO, THE SLOPE IS $m = \frac{1}{2}$ AND THE y-INTERCEPT IS $b = \frac{-2}{3}$.

EXAMPLE 9 WHAT IS THE EQUATION OF THE LINE PASSING THROUGH (–2, 0) AND (0, 5)?

SOLUTION: USING TWO-POINT FORM:

$$y - 0 = \frac{5 - 0}{0 - (-2)}(x + 2)$$

WHICH GIVES US, $5x - 2y + 10 = 0$ AS THE EQUATION OF THE LINE.

Exercise 4.5

1 FIND THE EQUATION OF THE LINE PASSING THROUGH THE GIVEN POINTS.

A A $(-2, -4)$ AND B $(-1, 5)$ **B** C $(2, -4)$ AND D $(-1, 5)$

C E $(3, 7)$ AND F $(8, 7)$ **D** G $(1, 1)$ AND H $(1 + \sqrt{2}, 1 - \sqrt{2})$

E P $(-1, 0)$ AND THE ORIGIN **F** Q $(4, -1)$ AND R $(4, -4)$

G M (,) AND N (3, –5) **H** T $\left(1\frac{1}{2}, -\frac{5}{2}\right)$ AND S $\left(-\frac{3}{2}, 1\right)$.

2 FIND THE EQUATION OF THE LINE WITH SLOPE m, PASSING THROUGH THE GIVEN POINT P.

A $m = \frac{3}{2}$; P $(0, -6)$ **B** $m = 0$; P $\left(\frac{}{2}, \frac{-}{4}\right)$

C $m = 1\frac{2}{3}$; P $(1, 1)$ **D** $m = -$; P $(0, 0)$

E $m = \sqrt{2}$; P $\left(\sqrt{2}, -\sqrt{2}\right)$ **F** $m = -1$; P $\left(\frac{1}{3}, \frac{3}{2}\right)$.

3 FIND THE EQUATION OF THE LINE WITH SLOPE m AND y-INTERCEPT b.

A $m = 0.1$; $b = 0$ **B** $m = -\sqrt{2}$; $b = -1$ **C** $m =$; $b = 2$

D $m = 1\frac{1}{3}$; $b = \frac{-5}{3}$ **E** $m = \frac{-1}{4}$; $b = 5$ **F** $m = \frac{2}{3}$; $b = 1.5$

4 SUPPOSE A LINE HAS x-INTERCEPT a AND y-INTERCEPT b, FOR $a, b \neq 0$; SHOW THAT THE EQUATION OF THE LINE IS $\frac{x}{a} + \frac{y}{b} = 1$.

5 FOR EACH OF THE FOLLOWING EQUATIONS, FIND THE SLOPE AND y-INTERCEPT:

A $\frac{3}{5}x - \frac{4}{5}y + 8 = 0$ **B** $-y + 2 = 0$ **C** $2x - 3y + 5 = 0$

D $x + \frac{1}{2}y - 2 = 0$ **E** $y + 2 = 2(x - 3y + 1)$.

6 A LINE PASSES THROUGH THE POINTS A (5, –1) AND B (–3, 3). FIND:

A THE POINT-SLOPE FORM OF THE EQUATION OF THE LINE.

B THE SLOPE-INTERCEPT FORM OF THE EQUATION OF THE LINE.

C THE TWO-POINT FORM OF THE EQUATION OF THE LINE. WHAT IS ITS GENERAL

7 FIND THE SLOPE AND y-INTERCEPT, IF THE EQUATION OF THE LINE IS:

A $\frac{1}{3}x - \frac{2}{3}y + 1 = y + x$ **B** $3(y - 2x) = y + \frac{1}{2}(1 - 2x)$.

8 A TRIANGLE HAS VERTICES AT A (–1, 1), B (1, 3) AND C (3, 1).

A FIND THE EQUATIONS OF THE LINES CONTAINING THE SIDES OF THE TRIANGLE

B IS THE TRIANGLE A RIGHT-ANGLED TRIANGLE?

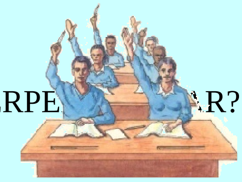
C WHAT ARE THE INTERCEPTS OF THE LINE PASSING THROUGH B AND C

4.4 PARALLEL AND PERPENDICULAR LINES

SLOPES CAN BE USED TO SEE WHETHER TWO NON-VERTICAL LINES IN A PLANE ARE PARALLEL, PERPENDICULAR, OR NEITHER.

FOR INSTANCE, THE LINES $y = x$ AND $y = x + 3$ ARE PARALLEL AND THE LINES $y = x$ AND $y = -x$ ARE PERPENDICULAR. HOW ARE THE SLOPES RELATED?

ACTIVITY 4.8



1 WHAT IS MEANT BY TWO LINES BEING PARALLEL? PERPENDICULAR?

2 IN FIGURE 4.19, ℓ_1 AND ℓ_2 ARE PARALLEL.

A CALCULATE THE SLOPE OF EACH LINE. **B** FIND THE EQUATION OF EACH LINE.

C DISCUSS HOW THEIR SLOPES ARE RELATED.

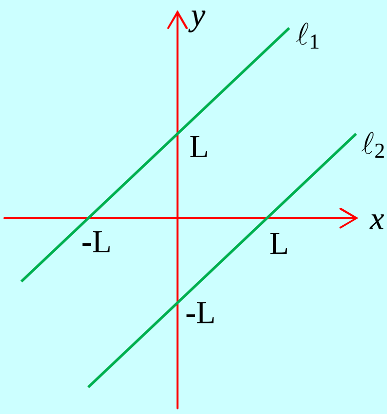


Figure 4.19

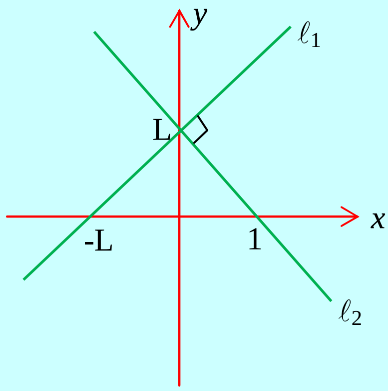


Figure 4.20

3 IN FIGURE 4.20 ABOVE, ℓ_1 AND ℓ_2 ARE PERPENDICULAR.

A CALCULATE THE SLOPE OF EACH LINE. **B** FIND THE EQUATION OF EACH LINE.

C DISCUSS HOW THEIR SLOPES ARE RELATED.

Theorem 4.1

If two non-vertical lines ℓ_1 and ℓ_2 are parallel to each other, then they have the same slope.

SUPPOSE YOU HAVE TWO NON-VERTICAL LINES ℓ_1 AND ℓ_2 WITH SLOPES m_1 AND m_2 AND INCLINATIONS θ_1 AND θ_2, RESPECTIVELY AS SHOWN IN FIGURE 4.21.

IF ℓ_1 IS PARALLEL TO ℓ_2, THEN $\theta_1 = \theta_2$ (WHY?)

CONSEQUENTLY, m_1 = TAN θ_1 = TAN θ_2 = m_2

State and prove the converse of the above theorem.

What can be stated for two vertical lines? Are they parallel?

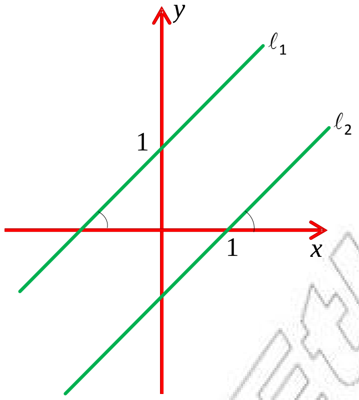


Figure 4.21

EXAMPLE 1 SHOW THAT THE LINE PASSING THROUGH A (−1, −1) AND B (2, −3) IS PARALLEL TO THE LINE PASSING THROUGH P (−3, −2) AND Q (3, −6).

SOLUTION: SLOPE OF $\overleftrightarrow{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-(-1)}{2-(-1)} = \frac{-3+1}{2+1} = -\frac{2}{3}$

SLOPE $\overleftrightarrow{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6-(-2)}{3-(-3)} = \frac{-6+2}{3+3} = -\frac{2}{3}$

SINCE $\overleftrightarrow{AB}$ AND $\overleftrightarrow{PQ}$ HAVE THE SAME SLOPE, $\overleftrightarrow{AB}$ IS PARALLEL TO $\overleftrightarrow{PQ}$, I.E. $\overleftrightarrow{AB} // \overleftrightarrow{PQ}$

RECALL THAT TWO LINES ARE PERPENDICULAR, IF THEY FORM A RIGHT-ANGLE AT THEIR INTERSECTION.

Theorem 4.2

Two non-vertical lines having slopes m_1 and m_2 are perpendicular, if and only if $m_1 \cdot m_2 = -1$.

Proof: SUPPOSE ℓ_1 IS PERPENDICULAR TO ℓ_2.

Note: IF ONE OF THE LINES IS A VERTICAL LINE, THEN THE OTHER ONE MUST BE A HORIZONTAL LINE WHICH HAS SLOPE ZERO. SO, ASSUME THAT NEITHER LINE IS VERTICAL.

LET m_1 AND m_2 BE THE SLOPES OF ℓ_1 AND ℓ_2, RESPECTIVELY.

LET R(x_0, y_0) BE THE POINT OF INTERSECTION AND CHOOSE P(x_1, y_1) AND Q(x_2, y_2) ON ℓ_1 AND ℓ_2, RESPECTIVELY.

DRAW TRIANGLES QSR AND RTP AS SHOWN IN FIGURE 4.22.

ΔQSR AND ΔRTP ARE SIMILAR, (WHY?)

$$\frac{PT}{RT} = \frac{RS}{QS} \quad \text{(WHY?)}$$

$$\frac{y_1 - y_o}{x_1 - x_o} = \frac{x_o - x_2}{y_2 - y_o} = -\left(\frac{x_2 - x_o}{y_2 - y_o}\right)$$

$$\frac{y_1 - y_o}{x_1 - x_o} = \frac{-1}{\frac{y_2 - y_o}{x_2 - x_o}}$$

$$m_1 = -\frac{1}{m_2} \text{ OR } m_1 m_2 = -1$$

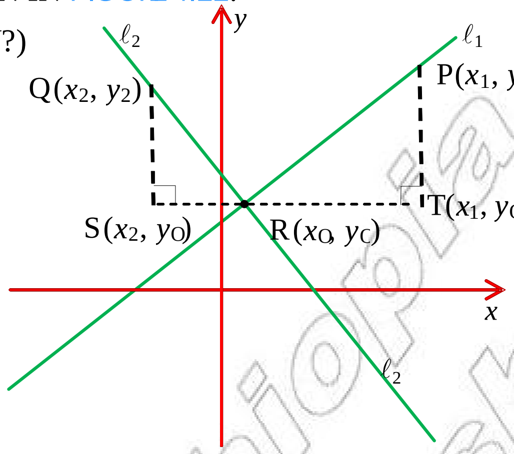


Figure 4.22

AS AN EXERCISE, START WITH $\frac{QS}{RS} = \frac{RT}{PT}$ AND CONCLUDE THAT $m_2 = -\frac{1}{m_1}$.

CONVERSELY, YOU COULD SHOW THAT IF TWO LINES HAVE SLOPES m_1 AND m_2 WITH $m_1 m_2 = -1$, THEN THE LINES ARE PERPENDICULAR. THIS CAN BE DONE BY REVERSING THE ABOV CONCLUDING THAT THE TWO TRIANGLES ARE SIMILAR. COMPLETE THE PROOF.

EXAMPLE 2 SUPPOSE ℓ_1 PASSES THROUGH P (–1, –3) AND Q (2, 6). FIND THE SLOPE m_2 OF ANY LINE ℓ_2 THAT IS:

A PARALLEL TO ℓ_1 **B** PERPENDICULAR TO ℓ_1

SOLUTION: THE SLOPE OF ℓ_1 IS

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-(-3)}{2-(-1)} = \frac{9}{3} = 3. \text{ SO,}$$

A THE SLOPE OF LINE ℓ_2 PARALLEL TO ℓ_1 IS $m_1 = 3$

B THE SLOPE OF LINE ℓ_2 PERPENDICULAR TO ℓ_1 IS $m_2 = -\frac{1}{m_1} = -\frac{1}{3}$

EXAMPLE 3 FIND THE EQUATION OF THE LINE PASSING THROUGH P(5, –1) AND PERPENDICULAR TO THE LINE $x - 3y = -7$.

SOLUTION: FROM $x - 3y = -7$, $y = \frac{1}{3}x + \frac{7}{3}$ SO, $m_1 = \frac{1}{3}$

LET THE SLOPE OF THE REQUIRED LINE BE m_2. THEN, $m_1 m_2 = -1$ GIVES $m_2 = \frac{-1}{m_1} = -3$

THEREFORE, THE REQUIRED EQUATION OF THE LINE IS $y + 1 = -3(x - 5)$ I.E. $y = -3x + 14$.

Exercise 4.6

1 In each of the following, determine whether the line through A and B is parallel to or perpendicular to the line through P and Q:

a A (−1, 3) and B (2, −2)
P (1, 4) and Q (−2, 9)

b A (−3, 5) and B (2, −5)
P (−1, 4) and Q (1, 5).

2 Find the slope of the line that is perpendicular to the line joining P (−3, 2) and Q (–3, –2).

3 Use slope to show that the quadrilateral ABCD with vertices A (−5, −2), B (−3, 1), C (3, 0) and D (1, −3) is a parallelogram.

4 Let ℓ be the line with equation $2x - 3y = 6$. Find the slope-intercept form of the equation of the line that passes through the point P (2, −1) and is:

a parallel to ℓ **b** perpendicular to ℓ

5 Find the equation of a line passing through point P and parallel to the line ℓ for:

a $\ell: 2x - 5y - 4 = 0$; P (–1, 2) **b** $\ell: 3x + 6 = 0$; P (4, –6).

6 Determine which of the following pairs of lines whose equations are given are perpendicular or parallel or neither:

A $3x - y + 5 = 0$ and $x + 3y - 1 = 0$

B $3x - 4y + 1 = 0$ and $4x - 3y + 1 = 0$

C $4x - 10y + 8 = 0$ and $10x + 6y - 3 = 0$

D $2x + 2y = 4$ and $x + y = 10$.

7 Find the equation of the line passing through the point (2, 6) and:

A parallel to the line passing through the points A (3, −1) and B (−1, 3)

B parallel to the line $\ell: x + y = 2$

C perpendicular to the line joining the points A (−1, 2) and B (4, −2)

D perpendicular to the line $\ell: y = x + 1$.

8 Determine k so that the line with equation $4x + ky = 12$ will be:

A parallel to the line with equation $x = 3y$

B perpendicular to the line with equation $x - 3y = 5$

9 Show that the plane figure with vertices:

A A (6, 1), B (5, 6), C (−4, 3) and D (−3, −2) is a parallelogram

B A (2, 4), B(1, 5), C (−2, 2) and D (−1, 1) is a rectangle.

10 The vertices of a triangle are A (−2, 5), B (3, 8) and C (6, −4). Show that the line joining the mid-points of sides $\overline{AB}$ and $\overline{BC}$ is parallel to and one-half the length of side $\overline{AC}$.

Key Terms

analytic geometry	general equation of a line	point-slope form
angle of inclination	horizontal line	slope (gradient)
coordinate geometry	inclination of a line	slope-intercept form
coordinates	mid-point	steepness
equation of a line	non-vertical line	two-point form

Summary

1 IF A POINT P HAS COORDINATES (a, b), THEN THE NUMBER a IS CALLED THE ***x*-coordinate** OR **abscissa** OF P AND b IS CALLED THE ***y*-coordinate** OR **ordinate** OF P.

2 THE **distance d** BETWEEN POINTS P (x_1, y_1) AND Q (x_2, y_2) IS GIVEN BY THE FORMULA

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

3 THE POINT R (x_0, y_0) DIVIDING THE LINE SEGMENT PQ **internally, in the ratio $m:n$** IS GIVEN BY

$$R\ (x_0, y_0) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right),$$

WHERE P (x_1, y_1) AND Q (x_2, y_2) ARE THE END-POINTS.

4 THE **mid-point** OF A LINE SEGMENT WHOSE END-POINTS ARE P (x_1, y_1) AND Q (x_2, y_2) IS GIVEN BY

$$M\ (x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

5 IF P (x_1, y_1) AND Q (x_2, y_2) ARE POINTS ON A LINE WITH $x_1 \neq x_2$, THEN THE **slope** (**gradient**) OF THE LINE IS GIVEN BY

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

6 IF   IS THE ANGLE BETWEEN THE POSITIVE x-AXIS AND THE LINE PASSING THROUGH THE POINTS P (x_1, y_1) AND Q (x_2, y_2), $x_1 \neq x_2$, THEN THE **slope** OF THE LINE IS GIVEN BY

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \text{TAN}$$

7 THE GRAPH OF THE EQUATION $x = c$ IS THE **vertical line** THROUGH P $(c, 0)$ AND HAS NO SLOPE.

8 THE **equation of the line** WITH SLOPE m AND PASSING THROUGH THE POINT P (x_1, y_1) IS GIVEN BY

$$y - y_1 = m\,(x - x_1)$$

9 THE EQUATION OF THE LINE WITH SLOPE m AND y-INTERCEPT b IS GIVEN BY

$$y = mx + b$$

10 THE EQUATION OF THE LINE PASSING THROUGH POINTS P(x_1, y_1) AND Q(x_2, y_2) IS GIVEN BY

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),\ x_1 \neq x_2$$

11 THE GRAPH OF EVERY FIRST DEGREE (LINEAR) EQUATION $Ax + By + C = 0$, $A, B \neq 0$ IS A ***straight line*** AND EVERY STRAIGHT LINE IS A GRAPH OF A FIRST DEGREE EQUATION.

12 TWO NON-VERTICAL LINES ARE ***parallel***, IF AND ONLY IF THEY HAVE THE SAME SLOPE.

13 LET ℓ_1 BE A LINE WITH SLOPE m_1 AND ℓ_2 BE A LINE WITH SLOPE m_2. THEN ℓ_1 AND ℓ_2 ARE ***perpendicular*** LINES IF AND ONLY IF $m_1 \cdot m_2 = -1$.

Review Exercises on Unit 4

1 SHOW THAT THE POINTS A (1, 1), B (−1, 1) AND C$(\sqrt{3}, \sqrt{3})$ ARE THE VERTICES OF AN EQUILATERAL TRIANGLE.

2 FIND THE COORDINATES OF THE THREE POINTS THAT DIVIDE THE LINE SEGMENT P (−4, 7) AND Q (10, −9) INTO FOUR PARTS OF EQUAL LENGTH.

3 FIND THE EQUATION OF THE LINE WHICH PASSES THROUGH POINTS P (−4, −2) AND Q (3, 6).

4 FIND THE EQUATION OF THE LINE:

A WITH SLOPE −3 THAT PASSES THROUGH P (8, 3).

B WITH SLOPE $-\frac{1}{2}$ THAT PASSES THROUGH Q (−2, 5).

5 IN EACH OF THE FOLLOWING, SHOW THAT THE THREE POINTS ARE VERTICES OF A RIGHT ANGLED TRIANGLE:

A A (0, 0), B (1, 1), C (2, 0) **B** P (3, 1), Q (−3, 4), R (−3, 1).

6 FIND THE SLOPE AND y-INTERCEPT OF THE LINE WITH THE FOLLOWING EQUATIONS:

A $2x - 3y = 4$ **B** $2y - 5x - 2 = 0$

C $5y + 6x - 4 = 0$ **D** $3y = 7x + 1$.

7 FIND THE EQUATION OF THE STRAIGHT LINE PASSING THROUGH P (−2, 1) AND:

A PARALLEL TO THE LINE WITH EQUATION $-3y = 12x$

B PERPENDICULAR TO THE LINE WITH EQUATION $6x + 5y = 10$.

8 LET ℓ BE THE LINE THROUGH A (−4, 5) AND B (3, t) THAT IS PERPENDICULAR TO THE LINE THROUGH P (1, 3) AND Q (−4, 2). FIND THE VALUE OF t

9 LET ℓ BE THE LINE THROUGH A (−4, 3) AND B (t, −2) THAT IS PARALLEL TO THE LINE THROUGH P (−2, 4) AND Q (4, −1). FIND THE VALUE OF t

10 PROVE THAT THE CONDITION FOR LINES $Ax + By + C = 0$ AND $ax + by + c = 0$ TO BE PERPENDICULAR MAY BE WRITTEN IN THE FORM

$$Aa + Bb = 0,\ \text{WHERE } B, b \neq 0.$$